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Pancyclicity of k-ary n-cube networks with faulty vertices and edges[☆]

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ABSTRACT

A graph G is said to be f-fault p-pancyclic if after removing f faulty vertices and/or edges from G, the resulting graph contains a cycle of every length from p to $|V(G)|$. In this paper, we consider one of the most popular networks which is named k-ary n-cube, and show that it is $(2n-2)$-fault k-pancyclic if $k \geq 3$ is odd. Finally, an example shows that our result is best possible in some sense.

1. Introduction

Network topology is a crucial factor for an interconnection network since it determines the performance of the network and the distributed systems. The network topology considered in this paper is the k-ary n-cube, denoted by Q_n^k, which has been proved to possess many attractive properties such as regularity, node transitivity and edge transitivity. Moreover, many interconnection networks can be viewed as the subclasses of Q_n^k include the cycle, the torus and the hypercube. Massively parallel systems have been built with a k-ary n-cube forming the underlying topology, such as the Cray T3D [7] and the iWarp [13].

Recently, the problem of embedding a guest interconnection network into a host interconnection network attracts much attention. That a host interconnection network can embed another guest interconnection network implies that the algorithms on the guest can be simulated on the host systematically [6,8].

The ring (cycle) is a popular interconnection network owing to its simple structure and low degrees. Moreover, many parallel algorithms have been devised on them [2,8,12]. The architecture of an interconnection network is usually represented by a graph $G = (V(G), E(G))$. Throughout this paper, we use network and graph, node and vertex, and link and edge, interchangeably. A graph G is said to be *p-pancyclic* if it contains a cycle of every length from p to $|V(G)|$ in G. When $p = 3$, the p-pancyclicity is just the classical pancyclicity, and when $p = |V(G)|$, the p-pancyclicity degenerates into the hamiltonicity. The p-pancyclicity is an important measurement to determine if a network topology is suitable for an application where mapping cycles of any length into the network topology is required. Many researchers have discussed how to embed cycles into various interconnection networks [4,5,14,15,18,17].

Since failures are inevitable, fault-tolerance is an important issue in multiprocessor systems. A graph G is f-fault p-pancyclic if $G - F$ still contains a cycle of every length from p to $|V(G - F_v)|$ for any $F \subseteq E(G) \cup V(G)$ with $|F| \leq f$, where $F_v \subseteq F$ is the set of faulty nodes. The f-fault hamiltonicity is defined analogously. The problem of cycle embedding

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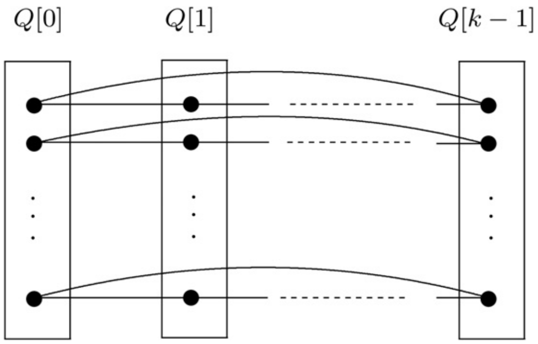

Fig. 1. Partition Q_n^k into $Q[0], Q[1], \ldots, Q[k-1]$.

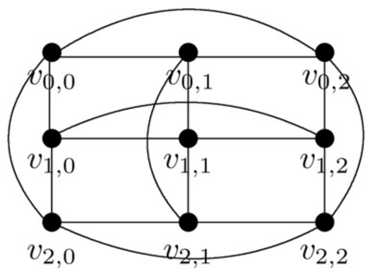

Fig. 2. Q_2^3.

in faulty k-ary n-cubes has been studied in [1,3,4,10,9,11,16,19]. In particular, Yang [19] et al. studied the hamiltonicity of k-ary n-cubes with both faulty nodes and edges. They presented the following result.

Theorem 1.1 *([19]). Q_n^k is $(2n-2)$-fault Hamiltonian for $n \geq 2$ and odd $k \geq 3$.*

In [3], Dong et al. determined the $(2n-2)$-fault pancyclicity of the 3-ary n-cube.

Theorem 1.2 *([3]). Q_n^3 is $(2n-2)$-fault pancyclic for $n \geq 2$.*

In this paper, we will generalize and extend Theorems 1.1 and 1.2, and show that Q_n^k is $(2n-2)$-fault k-pancyclic for $n \geq 2$ and odd $k \geq 3$. We prove the result by induction on n. In Section 3, we first verify the basis of the induction and show that Q_2^k is 2-fault k-pancyclic. In Section 4 we deal with the induction step to complete the proof. Section 5 closes the work.

2. Terminology and notation

The k-ary n-cube Q_n^k ($n \geq 2$, $k \geq 3$) is a graph consisting of k^n vertices, each of which has the form $u = u_{n-1}u_{n-2}\ldots u_0$, where $0 \leq u_i \leq k-1$ for $0 \leq i \leq n-1$. Two vertices $u = u_{n-1}u_{n-2}\ldots u_0$ and $v = v_{n-1}v_{n-2}\ldots v_0$ are adjacent if and only if there exists an integer j, $0 \leq j \leq n-1$, such that $u_j = v_j \pm 1 \pmod k$ and $u_i = v_i$, for every $i \in \{0, 1, \ldots, j-1, j+1, \ldots, n-1\}$. Such an edge (u, v) is called a *j-dimensional* edge.

We can partition Q_n^k along the dimension j, by deleting all the j-dimensional edges, into k disjoint subcubes, $Q_n^k[0], Q_n^k[1], \ldots, Q_n^k[k-1]$ (abbreviated as $Q[0], Q[1], \ldots, Q[k-1]$, if there are no ambiguities; see Fig. 1). It is clear that each $Q[i]$, $0 \leq i \leq k-1$, is isomorphic to Q_{n-1}^k. For $p, q \in [0, k-1]$, we use $Q[p, q]$ to denote the subgraph of Q_n^k which is induced by $\{u : u \in V(Q[i]), i = p, p+1, \ldots, q-1, q\}$. Then $Q[p, p] = Q[p]$.

For a faulty subset $F \subseteq V(Q_n^k) \cup E(Q_n^k)$, $Q_n^k - F$ denotes the subgraph obtained from Q_n^k by deleting all vertices (and their incident edges) and edges (without their incident vertices) of F. Denote $F_v = F \cap V(Q_n^k)$ and $F_e = F \cap E(Q_n^k)$. Furthermore, $F^{p:q}$ is the set of faulty elements in $Q[p, q]$. If $p = q$, write F^p for $F^{p:p}$. $F_v^{p:q}$, $F_e^{p:q}$, F_v^p and F_e^p are defined analogously. The set of faulty j-dimensional edges between $Q[a]$ and $Q[b]$ is denoted by $\mathcal{F}_j^{a,b}$.

3. K-pancyclicity of the faulty k-ary 2-cube networks

In this section, we verify the basis of the induction and show that Q_2^k is 2-fault k-pancyclic for $k \geq 3$ is odd.

We consider Q_2^k as the Cartesian product of two cycles of length k and we think of a vertex $v_{i,j}$ as indexed by its row i and column j. Fig. 2 shows Q_2^3. Given two row indices $i, j \in \{0, 1, \ldots, k-1\}$, where $j \neq i$, we define Row$(i : j)$ to be the

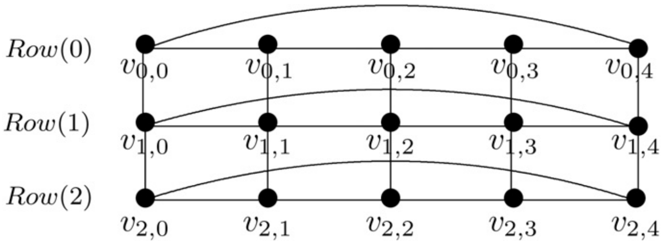

Fig. 3. Row$(0:2)$ of Q_2^5.

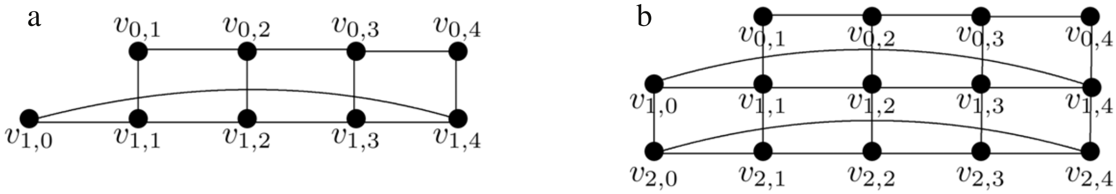

Fig. 4. (a) $R^*(2, 5)$ and (b) $R^*(3, 5)$.

subgraph of Q_2^k induced by the vertices on rows $i, i+1, \ldots, j$, if $i < j$, or rows $i, i+1, \ldots, k-1, 0, \ldots, j$, if $j < i$. In addition, Row$(i:i)$ is denoted by Row(i). Fig. 3 shows Row$(0:2)$ of Q_2^5.

The following lemma is from [11].

Lemma 3.1 *([11]). In* Row$(i, i+1)$ *of* Q_2^k*, every edge in* Row(i) *and* Row$(i+1)$ *lies in a cycle of every length from* k *to* $2k$*, where* $k \geq 3$ *is odd and* $0 \leq i \leq k-1$.

For convenience, when $k \geq 5$, we denote $R^*(m, k) = \text{Row}(0 : m-1) - v_{0,0}$, where $2 \leq m \leq k-1$. Fig. 4 shows $R^*(2, 5)$ and $R^*(3, 5)$. We first give some properties of $R^*(m, k)$.

In $R^*(m, k)$, let $M = \{(v_{1,i}, v_{1,i+1}) : i = 1, 3, \ldots, k-2\}$. For each edge $(v_{1,i}, v_{1,i+1}) \in M$, there is a 4-cycle $C_i = \langle v_{1,i}, v_{1,i+1}, v_{0,i+1}, v_{0,i}, v_{1,i}\rangle$ containing this edge. For any subset M' of M, we denote $\mathcal{C}(M') = \{C_i : (v_{1,i}, v_{1,i+1}) \in M'\}$. Obviously, $\mathcal{C}(M')$ is a graph consisting of $|M'|$ disjoint 4-cycles.

Let G_1 and G_2 be two graphs. Then $G_1 \Delta G_2$ is the graph induced by the edges of $E(G_1)\Delta E(G_2)$, where $E(G_1)\Delta E(G_2)$ denotes the symmetric difference of $E(G_1)$ and $E(G_2)$.

Lemma 3.2. *In* $R^*(2, k)$*, the edge* $(v_{1,2}, v_{1,3})$ *lies in a cycle of every length from* k *to* $2k-1$.

Proof. Clearly, Row(1) is exactly the cycle of length k that contains $(v_{1,2}, v_{1,3})$. Set $M = \{(v_{1,i}, v_{1,i+1}) : i = 1, 3, \ldots, k-2\}$. Note that $(v_{1,2}, v_{1,3}) \notin M$. Since $|M| = (k-1)/2$, for every $1 \leq l \leq (k-1)/2$, there exists a subset M' of M such that $|M'| = l$. It can be seen that Row$(1)\Delta\mathcal{C}(M')$ is a cycle of length $k+2l \in \{k+2, k+4, \ldots, 2k-1\}$ that contains $(v_{1,2}, v_{1,3})$.

We next find cycles of lengths $k+1, k+3, \ldots, 2k-2$ that contains $(v_{1,2}, v_{1,3})$. Let $H = R^*(2, k) - v_{1,0}$. It is easy to verified that the edge $(v_{1,2}, v_{1,3})$ lies in cycles of H of lengths $4, 6, 8, \ldots, 2k-2$. As $k+1 \geq 6$, the lemma follows. □

Lemma 3.3. *In* $R^*(3, k)$*, every edge of* Row(2) *lies in a cycle of every length from* k *to* $3k-1$.

Proof. Let e be an edge of Row(2). It follows from Lemma 3.1 that e lies in a cycle of Row$(1:2)$ of every length from k to $2k$. It suffices to show that e lies in a cycle of every length l with $l \in \{2k+1, 2k+2, \ldots, 3k-1\}$.

Case 1. $e \neq (v_{2,2}, v_{2,3})$.

Observe that Row$(0:1) - v_{0,0}$ is isomorphic to $R^*(2, k)$. By Lemma 3.2, the edge $(v_{1,2}, v_{1,3})$ lies in a cycle C of Row$(0:1) - v_{0,0}$ of every length l_0 with $l_0 \in \{k+1, k+2, \ldots, 2k-1\}$. Let $C' = \langle v_{1,2}, v_{1,3}, v_{2,3}, v_{2,2}, v_{1,2}\rangle$. It can be seen that Row$(2)\Delta C'\Delta C$ is the desired cycle of length l with $l = k + l_0 \in \{2k+1, 2k+2, \ldots, 3k-1\}$.

Case 2. $e = (v_{2,2}, v_{2,3})$.

Let $M = \{(v_{1,i}, v_{1,i+1}) : i = 1, 3, \ldots, k-2\}$ and let

$$C_1 = \langle v_{1,0}, v_{1,1}, \ldots, v_{1,k-1}, v_{2,k-1}, v_{2,k-2}, \ldots, v_{2,0}, v_{1,0}\rangle,$$

$$C_2 = \langle v_{1,0}, v_{2,0}, v_{2,1}, v_{1,1}, v_{1,2}, v_{2,2}, v_{2,3}, \ldots, v_{2,k-2}, v_{1,k-2}, v_{1,k-1}, v_{1,0}\rangle.$$

Then C_1 is a Hamiltonian cycle of Row$(1:2)$ of length $2k$ and C_2 is a Hamiltonian cycle of Row$(1:2) - v_{2,k-1}$ of length $2k-1$. Furthermore, both C_1 and C_2 contain e and all edges in M.

Since $|M| = (k-1)/2$, for every $1 \leq l' \leq (k-1)/2$, there exists a subset M' of M such that $|M'| = l'$. It can be seen that $C_1\Delta\mathcal{C}(M')$ is a cycle of every length l with $l = 2k + 2l' \in \{2k+2, 2k+4, \ldots, 3k-1\}$ and $C_2\Delta\mathcal{C}(M')$ is a cycle of every length l with $l = 2k - 1 + 2l' \in \{2k+1, 2k+3, \ldots, 3k-2\}$. The proof is complete. □

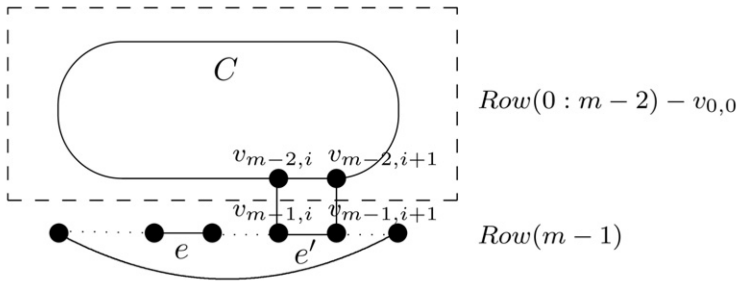

Fig. 5. An illustration for the proof in Lemma 3.4.

Lemma 3.4. *In $R^*(m, k)$, if $m \geq 3$, then every edge in* $\mathrm{Row}(m-1)$ *lies in a cycle of every length from k to $mk-1$.*

Proof. We proof this lemma by induction on m. When $m = 3$, the lemma is true by Lemma 3.3. Assume that $m \geq 4$ and the lemma is true for $R^*(m-1, k)$. Then every edge in $\mathrm{Row}(m-2)$ lies in a cycle of $\mathrm{Row}(0 : m-2) - v_{0,0}$ of every length from k to $(m-1)k-1$. Let e be an edge of $\mathrm{Row}(m-1)$. We will show that e lies in a cycle of $R^*(m, k)$ of every length from k to $mk-1$.

By Lemma 3.1, e lies in a cycle of $\mathrm{Row}(m-2 : m-1)$ of every length from k to $2k$. Next, choose in $\mathrm{Row}(m-1)$ an edge e' such that $e' \neq e$. See Fig. 5. Suppose that $e' = (v_{m-1,i}, v_{m-1,i+1})$. By the induction hypothesis, the edge $(v_{m-2,i}, v_{m-2,i+1})$ lies in a cycle C of $\mathrm{Row}(0 : m-2) - v_{0,0}$ of every length from $k+1$ to $(m-1)k-1$. Let $C' = \langle v_{m-2,i}, v_{m-2,i+1}, v_{m-1,i+1}, v_{m-1,i}, v_{m-2,i}\rangle$. Then $\mathrm{Row}(m-1)\Delta C'\Delta C$ is the desired cycle of every length $l = |E(\mathrm{Row}(m-1))| + |E(C)| \in \{2k+1, 2k+2, \ldots, mk-1\}$. □

Lemma 3.5. *Let F be a set of 2 faulty vertices in Q_2^k, where $k \geq 5$ is odd. Then $Q_2^k - F$ contains two disjoint subgraphs such that one is isomorphic to $R^*(p, k)$ and the other is isomorphic to $R^*(q, k)$, where $p > q \geq 2$ and $p + q = k$.*

Proof. By symmetry, we may assume that one faulty vertex is $v_{0,0}$ and the other is $v_{i,j}$, where $0 \leq j \leq i \leq k-1$. Clearly, $i \neq 0$, otherwise, $j = i = 0$ and so $v_{i,j} = v_{0,0}$.

If $i = 1$, then the faulty vertex $v_{i,j}$ is in $\mathrm{Row}(1)$. Let $H_1 = \mathrm{Row}(1 : k-2) - v_{i,j}$ and $H_2 = \mathrm{Row}(k-1 : 0) - v_{0,0}$. If $i = k-1$, let $H_1 = \mathrm{Row}(0 : k-3) - v_{0,0}$ and $H_2 = \mathrm{Row}(k-2 : k-1) - v_{i,j}$. It can be verified that H_1 is isomorphic to $R^*(k-2, k)$ and H_2 is isomorphic to $R^*(2, k)$. Moreover, $p = k-2 > 2 = q$ and $p + q = k$.

If $2 \leq i \leq (k-1)/2$, let $H_1 = \mathrm{Row}(i : k-1) - v_{i,j}$ and $H_2 = \mathrm{Row}(0 : i-1) - v_{0,0}$. Then H_1 is isomorphic to $R^*(k-i, k)$ and H_2 is isomorphic to $R^*(i, k)$. Moreover, $p = k-i > i = q \geq 2$ and $p + q = k$.

If $(k+1)/2 \leq i \leq k-2$, let $H_1 = \mathrm{Row}(0 : i-1) - v_{0,0}$ and $H_2 = \mathrm{Row}(i : k-1) - v_{i,j}$. Then H_1 is isomorphic to $R^*(i, k)$ and H_2 is isomorphic to $R^*(k-i, k)$. Moreover, $p = i > k-i = q \geq 2$ and $p + q = k$. The proof is complete. □

Lemma 3.6. *Q_2^k with 2 faulty vertices is k-pancyclic for odd $k \geq 3$.*

Proof. Since the case that $k = 3$ has been handled by Theorem 1.2, we consider $k \geq 5$. By Lemma 3.5, the faulty Q_2^k has two disjoint subgraphs H_1 and H_2 such that H_1 is isomorphic to $R^*(p, k)$ and H_2 is isomorphic to $R^*(q, k)$, where $p > q \geq 2$ and $p + q = k$. We will find a cycle of the faulty Q_2^k of every length l with $k \leq l \leq k^2 - 2$.

Case 1. $k \leq l \leq qk$.

As $p > q \geq 2$, we have $p \geq 3$. By Lemma 4.4, in H_1 there is a cycle of every length from k to $pk-1$. Note that $k < qk < pk-1$. Therefore, the desired cycle of length l with $k \leq l \leq qk$ exists.

Case 2. $qk + 1 \leq l \leq k^2 - 2$.

Due to the symmetry, we may assume that $H_1 = \mathrm{Row}(0 : p-1) - v_{0,0}$ and $H_2 = \mathrm{Row}(p : k-1) - v_{i,j}$. Note that vertex $v_{i,j}$ is on $\mathrm{Row}(p)$ or $\mathrm{Row}(k-1)$.

As $q \geq 2$, we have $(q-1)k + 1 > k$. As H_2 is isomorphic to $R^*(q, k)$, by Lemmas 3.2 and 3.4, in H_2 there is a cycle C_0 of length l_0 with $(q-1)k+1 \leq l_0 \leq qk-1$. Noting that $l_0 \geq (q-1)k+1$, we see that the cycle C_0 contains at least one vertex v of $\mathrm{Row}(p)$. Observe that v is incident to at most 3 edges of H_2 and only one edge is not in $\mathrm{Row}(p)$. Thus C_0 must contain one edge of $\mathrm{Row}(p)$ which is incident to v. Assume that this edge is $(v_{p,r}, v_{p,r+1})$.

As H_1 is isomorphic to $R^*(p, k)$ and $p \geq 3$, by Lemma 3.4, the edge $(v_{p-1,r}, v_{p-1,r+1})$ lies in a cycle C_1 of H_1 of length l_1 with $k \leq l_1 \leq pk-1$. Let $C' = \langle v_{p-1,r}, v_{p-1,r+1}, v_{p,r+1}, v_{p,r}, v_{p-1,r}\rangle$. It can be seen that $C_0\Delta C'\Delta C_1$ is the desired cycle of every length l with $qk + 1 \leq l = l_0 + l_1 \leq k^2 - 2$. □

Lemma 3.7. *Q_2^k is 2-fault k-pancyclic for odd $k \geq 3$.*

Proof. In order to prove the assertion, our discussion will proceed by distinguishing the following six cases.

Case 1. $|F_v| = 2$ and $|F_e| = 0$.

This case has been handled by Lemma 3.6.

Case 2. $|F_v| = 1$ and $|F_e| = 1$.

By Theorem 1.1, we get the cycle of length $k^2 - 1$. Temporarily regard one healthy vertex incident to the faulty edge as faulty. By Case 1, we have the cycle of every length from k to $k^2 - 2$.

Case 3. $|F_v| = 0$ and $|F_e| = 2$.

By Theorem 1.1, we get the cycle of length k^2. Temporarily regard one healthy vertex incident to one faulty edge as faulty. By Case 2, we have the cycle of every length from k to $k^2 - 1$.

Case 4. $|F_v| = 1$ and $|F_e| = 0$.

Temporarily regard one arbitrary healthy edge as faulty. By Case 2, we have the cycle of every length from k to $k^2 - 1$.

Case 5. $|F_v| = 0$ and $|F_e| = 1$.

Temporarily regard one arbitrary healthy edge as faulty. By Case 3, we have the cycle of every length from k to k^2.

Case 6. $|F_v| = |F_e| = 0$.

The k-pancyclicity of Q_2^k for this case has been shown in [14]. In fact, temporarily regard one arbitrary healthy edge as faulty. By Case 5, we have the cycle of every length from k to k^2. □

4. *K*-pancyclicity of the faulty *k*-ary *n*-cube networks

In this section, we deal with the inductive step and show that when $n \geq 3$, the faulty Q_n^k is $(2n-2)$-fault k-pancyclic. First assume that Q_{n-1}^k is $(2n-4)$-fault k-pancyclic. We will give some useful lemmas.

Let F ($|F| \leq 2n-2$) be the faulty set. Then we can divide Q_n^k into $Q[0], Q[1], \ldots, Q[k-1]$ along some dimension such that $|F^i| \leq 2n-3$ for every $0 \leq i \leq k-1$. Clearly, it is enough to consider $|F| = 2n-2$. If there is a faulty edge, we can divide Q_n^k along the dimension of this faulty edge. On the other hand, suppose that $F \subseteq V(Q_n^k)$. Since $|F| = 2n-2 \geq 4$, for every $n \geq 3$, picking arbitrarily two faulty vertices in Q_n^k, we can divide Q_n^k along some dimension such that these two faulty vertices are in different Q_{n-1}^k.

Let d be a dimension as above. If all faults are d-dimensional edges (that is, $|F| \neq 0$ and $|F^0| = |F^1| = \cdots = |F^{k-1}| = 0$), then we may assume without loss of generality that $\mathcal{F}_d^{0,1} \neq \emptyset$. Otherwise, we may assume that $|F^0| \geq |F^j|$ for every $j \in [1, k-1]$. Thus, we always have $|F^0| \geq |F^j|$ and $|F^{1:k-1}| \leq 2n-3$.

Throughout this section, if u is a vertex of $Q[i]$, then we often denote it by u^i, and we refer to its neighbor in $Q[j]$ as u^j. Let P be a path of $Q[i]$ with two ends u^i and v^i. We say that P is (i, j)-free if $\{(u^i, u^j), (v^i, v^j)\}$ is a pair of healthy edges.

Lemma 4.1 (*[19]*). *If $|F^i| \leq 2n-5$, then $Q[i] - F^i$ has a Hamiltonian path connecting any two healthy vertices.*

Lemma 4.2. *In $Q[0] - F^0$, there is a Hamiltonian path which is $(0, 1)$-free or $(0, k-1)$-free.*

Proof. Recall that $|F^0| \leq 2n-3$. There are three cases to consider.

Case 1. $|F^0| \leq 2n-5$.

As $n \geq 3$ and $k \geq 3$, we have

$$|V(Q[0])| - (|F_v^0| + |F_v^1| + |\mathcal{F}_d^{0,1}|) \geq |V(Q[0])| - |F| \geq k^{n-1} - (2n-2) > 2.$$

Thus, in $Q[0]$ there exist two healthy vertices u^0 and v^0 such that $\{(u^0, u^1), (v^0, v^1)\}$ is a pair of healthy edges. As $|F^0| \leq 2n-5$, Lemma 4.1 implies that $Q[0] - F^0$ has a Hamiltonian path connecting u^0 and v^0. The assertion holds.

Case 2. $|F^0| = 2n-4$.

As $|F^0| = 2n-4$, by the induction hypothesis, $Q[0] - F^0$ is k-pancyclic. So there is a Hamiltonian cycle C in $Q[0] - F^0$. As $|F| - |F^0| \leq 2$, there are at most two faults outside $Q[0]$. We may choose in C an edge (u^0, v^0) such that $\{(u^0, u^1), (v^0, v^1)\}$ is a pair of healthy edges because $|E(C)| = |V(Q_n^k)| - |F_v^0| \geq k^{n-1} - (2n-4) > 4$ and one fault outside $Q[0]$ blocks at most two edges in C. Therefore, the path $P = C - (u^0, v^0)$ is a required Hamiltonian path.

Case 3. $|F^0| = 2n-3$.

Let f be a fault of F^0 and let $F' = F^0 \setminus \{f\}$. Then $|F'| = 2n-4$ and so $Q[0] - F'$ has a Hamiltonian cycle C. If f is not in C, then, by a similar proof above, we may obtain a required Hamiltonian path. If f is in C, then the path $P = C - f$ is a Hamiltonian path of $Q[0] - F^0$. Let u^0, v^0 be the two ends of P. As there is at most one fault outside $Q[0]$, either $\{(u^0, u^1), (v^0, v^1)\}$ or $\{(u^0, u^{k-1}), (v^0, v^{k-1})\}$ is a pair of healthy edges. The proof of the lemma is complete. □

The following lemmas are from [11].

Lemma 4.3 (*[11]*). *Let $p, q \in [0, k-1]$. If $|F^{p:q}| \leq 2n-3$ and $|F^i| \leq 2n-5$ for every $i \in \{p, p+1, \ldots, q\}$, then for any two distinct vertices $s, t \in V(Q[p] - F^p)$, there is an s-t path of every length from $(n-1)(k-1) - 1$ to $(q-p+1) \times k^{n-1} - |F_v^{p:q}| - 1$ in $Q[p:q] - F^{p:q}$.*

Lemma 4.4 (*[11]*). *Let $p, q \in [0, k-1]$. If $|F^{p:q}| \leq 2n-3$ and $|F^i| \leq 2n-5$ for every $i \in \{p, p+1, \ldots, q\}$, then for any two distinct vertices $s \in V(Q[p] - F^p)$ and $t \in V(Q[r] - F^r)$, there is an s–t path of every length from $(n-1)(k-1) - 1 + 2(r-p)$ to $(q-p+1) \times k^{n-1} - |F_v^{p:q}| - 1$ in $Q[p:q] - F^{p:q}$, where $p \leq r \leq q$.*

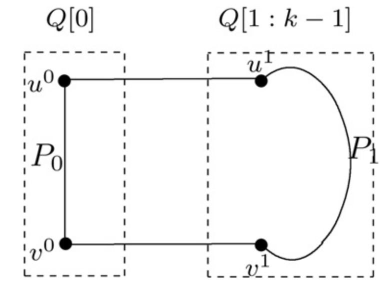

Fig. 6. A cycle of every length from $k^{n-1}-|F_v^0|+(n-1)(k-1)$ to $k^n-|F_v|$.

Lemma 4.5. *If $|F^j|\le 2n-5$ for every $1\le j\le k-1$, then Q_n^k-F is k-pancyclic.*

Proof. We find the desired cycle of every length l with $k\le l\le k^n-|F_v|$ in three cases.

Case 1. $k\le l\le k^{n-1}-|F_v^1|$.

As $|F^1|\le 2n-5<2n-4$, by the induction hypothesis, $Q[1]-F^1$ is k-pancyclic. The desired cycle of every length from k to $k^{n-1}-|F_v^1|$ inclusive can be found in $Q[1]-F^1$.

Case 2. $k^{n-1}-|F_v^1|+1\le l\le (k-1)\times k^{n-1}-|F_v^{1:k-1}|$.

Recalling that $|F^{1:k-1}|\le 2n-3$ and $|F^j|\le 2n-5$ for every $1\le j\le k-1$, Lemma 4.3 implies that $Q[1:k-1]-F^{1:k-1}$ has a cycle of every length from $(n-1)(k-1)$ to $(k-1)\times k^{n-1}-|F_v^{1:k-1}|$. As $n\ge 3$ and $k\ge 3$, we have

$$k^{n-1}-|F_v^1|+1>k^{n-1}-(2n-5)+1>(n-1)(k-1).$$

Thus the desired cycles can be found.

Case 3. $(k-1)\times k^{n-1}-|F_v^{1:k-1}|+1\le l\le k^n-|F_v|$.

By Lemma 4.2, $Q[0]-F^0$ has a Hamiltonian path P_0 which is (0, 1)-free or (0, $k-1$)-free. W.l.o.g., assume that P_0 is (0, 1)-free. Let u^0, v^0 be the two ends of P_0. Then $\{(u^0,u^1),(v^0,v^1)\}$ is a pair of healthy edges. See Fig. 6. Clearly, $|E(P_0)|=k^{n-1}-|F_v^0|-1$.

By Lemma 4.3, in $Q[1:k-1]$ there is a path P_1 between u^1 and v^1 of every length from $(n-1)(k-1)-1$ to $(k-1)\times k^{n-1}-|F_v^{1:k-1}|-1$. Therefore, $\langle u^0,P_0,v^0,v^1,P_1,u^1,u^0\rangle$ is a cycle of length l with $k^{n-1}-|F_v^0|+(n-1)(k-1)\le l=|E(P_0)|+|E(P_1)|+2\le k^n-|F_v|$. As

$$\begin{aligned}(k-1)\times k^{n-1}-|F_v^{1:k-1}|+1 &\ge 2\times k^{n-1}-(2n-3)+1\\ &> k^{n-1}+(n-1)(k-1)\quad (n\ge 3\text{ and }k\ge 3)\\ &> k^{n-1}-|F_v^0|+(n-1)(k-1),\end{aligned}\tag{4.1}$$

the desired cycles can be found. □

Lemma 4.6. *If there exists an integer $j\in\{1,2,\ldots,k-1\}$ such that $|F^j|\ge 2n-4$, then Q_n^k-F is k-pancyclic.*

Proof. Recall that $|F^0|\ge|F^j|$. As $|F^j|\ge 2n-4$, we have $2(2n-4)\le|F^0\cup F^j|\le|F|\le 2n-2$, which implies that $n\le 3$. Combining this with the fact that $n\ge 3$, we have $n=3$. Moreover, $|F|=2n-2=4$ and $|F^0|=|F^j|=2n-4=2$.

Since the case that $k=3$ has been handled by Theorem 1.2, we consider $k\ge 5$. By symmetry, we may assume that $1\le j\le (k-1)/2$. We will find the desired cycle of every length l with $k\le l\le k^3-|F_v|$ in four cases.

Case 1. $k\le l\le k^2$.

As $|F|=4$ and $|F^0|=|F^j|=2$, then $F=F^0\cup F^j$ and $|F^{j+1}|=0$. By the induction hypothesis, $Q[j+1]$ contains a cycle of length $l\in\{k,k+1,\ldots,k^2\}$.

Case 2. $k^2+1\le l\le (k-j)\times k^2-|F_v^j|$.

By Lemma 4.3, it is easy to see that $Q[j+1:k-1]$ has a cycle of every length from $2(k-1)$ to $(k-(j+1))\times k^2$. As $k^2+1>2(k-1)$, we may obtain the required cycle of length $l\in\{k^2+1,\ldots,(k-(j+1))\times k^2\}$.

As $|F^j|=2=2n-4$, by the induction hypothesis, $Q[j]$ has a Hamiltonian cycle C_1. See Fig. 7. Choose in C_1 an edge (u^j,v^j). Let $P_1=C_1-(u^j,v^j)$. Clearly, $|E(P_1)|=k^2-|F_v^j|-1$. As $F=F^0\cup F^j$, both (u^j,u^{j+1}) and (v^j,v^{j+1}) are healthy. By Lemma 4.3, in $Q[j+1:k-1]$ there is a path P_2 connecting u^{j+1} and v^{j+1} of every length from $2(k-1)-1$ to $(k-(j+1))\times k^2-1$. Observe that $\langle u^j,P_1,v^j,v^{j+1},P_2,u^{j+1},u^j\rangle$ is a cycle of length $l=|E(P_1)|+|E(P_2)|+2$ with $k^2-|F_v^j|+2(k-1)\le l\le (k-j)\times k^2-|F_v^j|$.

As $k\ge 5$ and $j\le (k-1)/2$, we have $k-(j+1)\ge k-(k+1)/2\ge 2$ and so

$$\begin{aligned}(k-(j+1))\times k^2+1 &\ge 2\times k^2+1\\ &\ge k^2-|F_v^j|+(k^2+1)\\ &> k^2-|F_v^j|+2(k-1).\end{aligned}\tag{4.2}$$

Therefore, we may obtain a cycle of length $l\in\{(k-(j+1))\times k^2+1,\ldots,(k-j)\times k^2-|F_v^j|\}$.

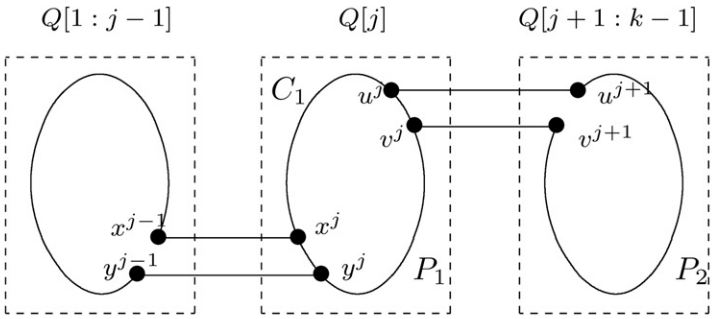

Fig. 7. An illustration for the proof in Cases 2 and 3 of Lemma 4.6.

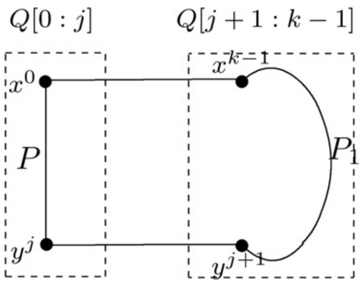

Fig. 8. A cycle of every length from $(j+1)\times k^2-|F_v|+4(k-1)-2j-2$ to $k^3-|F_v|$.

Case 3. $(k-j)\times k^2-|F_v^j|+1\le l\le (k-1)\times k^2-|F_v^j|$.

Note that this case is necessary only when $j\ge 2$. From Case 2, we see that in $Q[j:k-1]$ there is a cycle C of length from $k^2-|F_v^j|+2(k-1)$ to $(k-j)\times k^2-|F_v^j|$. Again see Fig. 7.

Choose in C an edge (x^j,y^j). Clearly, $\{(x^j,x^{j-1}),(y^j,y^{j-1})\}$ is a pair of healthy edges. By Lemma 4.3, in $Q[1:j-1]$ there is a path P_0 connecting x^{j-1} and y^{j-1} of every length from $2(k-1)-1$ to $(j-1)\times k^2-1$. Let $P=C-(x^j,y^j)$. Then $\langle x^j,P,y^j,y^{j-1},P_0,x^{j-1},x^j\rangle$ is a cycle of every length l with $k^2-|F_v^j|+4(k-1)\le l=|E(P)|+|E(P_0)|+2\le (k-1)\times k^2-|F_v^j|$. As

$$(k-j)\times k^2-|F_v^j|+1>2\times k^2-|F_v^j|\ge k^2-|F_v^j|+4(k-1),$$

we may obtain the desired cycle of length $l\in\{(k-j)\times k^2-|F_v^j|+1,\dots,(k-1)\times k^2-|F_v^j|\}$.

Case 4. $(k-1)\times k^2-|F_v^j|+1\le l\le k^3-|F_v|$.

We first claim that $Q[0:j]-F$ contains a Hamiltonian path with one end in $Q[0]$ and the other in $Q[j]$. As $|F^0|=|F^j|=2=2(n-1)-2$, Theorem 1.1 implies that $Q[0]-F^0$ ($Q[j]-F^j$) has a Hamiltonian cycle C_0 (C_j). Recall that $1\le j\le (k-1)/2$.

Suppose $j=1$. Choose in C_0 a vertex x^0 such that x^1 is a healthy vertex in $Q[1]$. Let y^0 be a neighbor of x^0 in C_0 and z^1 be a neighbor of x^1 in C_1. Combining $C_0-(x^0,y^0)$ with $C_1-(x^1,z^1)$ as well as the edge (x^0,x^1), we may obtain a required Hamiltonian path.

Suppose $2\le j\le (k-1)/2$. Choose in C_0 an edge (x^0,y^0). As $|F^1|=0$, vertex x^1 is healthy. Choose in C_j an edge (u^j,v^j). Clearly, either $u^{j-1}\ne x^1$ or $v^{j-1}\ne x^1$. W.l.o.g., assume that $u^{j-1}\ne x^1$. By Lemma 4.4, $Q[1:j-1]$ has a Hamiltonian path P_1 connecting x^1 and u^{j-1}. Combining $C_0-(x^0,y^0)$, $C_j-(u^j,v^j)$ and P_1 as well as two edges (x^0,x^1), (u^{j-1},u^j), we may obtain a required Hamiltonian path. The claim holds.

Let P be a Hamiltonian path of $Q[0:j]-F$ with two ends x^0 and y^j. Then $|E(P)|=(j+1)\times k^2-|F_v|-1$. See Fig. 8. By Lemma 4.4, $Q[j+1:k-1]$ has a path P_1 connecting y^{j+1} and x^{k-1} of every length from $2(k-1)-1+2(k-j-2)=4(k-1)-2j-3$ to $(k-(j+1))\times k^2-1$. Therefore, $\langle x^0,P,y^j,y^{j+1},P_1,x^{k-1},x^0\rangle$ is a cycle of every length l with $(j+1)\times k^2-|F_v|+4(k-1)-2j-2\le l=|E(P)|+|E(P_1)|+2\le k^3-|F_v|$.

As $k\ge 5$ and $j\le (k-1)/2$, we have $k-(j+1)\ge 2$ and so $k-1\ge j+2$. Therefore,

$$\begin{aligned}(k-1)\times k^2-|F_v^j|+1 &\ge (j+2)\times k^2-|F_v^j|+1\\ &\ge (j+1)\times k^2-|F_v|+(k^2+1)\\ &> (j+1)\times k^2-|F_v|+4(k-1)-4\\ &\ge (j+1)\times k^2-|F_v|+4(k-1)-2j-2,\end{aligned}\tag{4.3}$$

and we may obtain the desired cycle of length $l\in\{(k-1)\times k^2-|F_v^j|+1,\dots,k^3-|F_v|\}$. □

By Lemmas 4.5 and 4.6, we have our main result as follows.

Theorem 4.7. *Q_n^k is $(2n-2)$-fault k-pancyclic for $n \geq 3$ and $k \geq 3$ is odd.*

Finally we show that the bound $2n-2$ is optimal. Consider a vertex v of Q_n^k and destroy $2n-1$ neighbors of v. As v has $2n$ neighbors in total, then v is incident to only one healthy neighbor in the faulty Q_n^k. Obviously, there is no Hamiltonian cycle in the faulty Q_n^k. Therefore, the bound $2n-2$ is sharp.

5. Conclusions

The panconnectivity and pancyclicity problems are very attractive topics currently. In this paper, we give the k-pancyclicity of k-ary n-cubes with at most $2n-2$ faulty vertices and edges for $k \geq 3$ is odd. Note that when k is even, a k-ary n-cube is a bipartite graph and so it certainly cannot be pancyclic. Furthermore, there is no cycle of length more than $k^n - 2|F_v|$ if the faulty vertices are all in the same part, where F_v is the set of faulty vertices. Clearly, if faulty vertices exist, then $k^n - 2|F_v| < k^n - |F_v| = |V(Q_n^k - F)|$.

It may be of interest to explore further the pancyclicity of the faulty Q_n^k with more than $2n-2$ faulty elements. For investigating this problem, a significant amount of extra work has to be done.

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